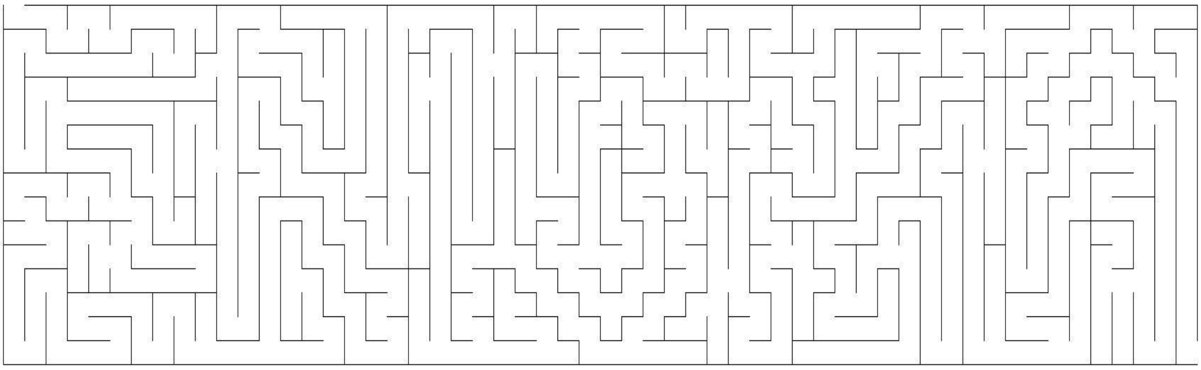
Enigma:
The Newsletter of PuzzleSIG



ENIGMA

252

'Mathematical problems, or puzzles, are important to real mathematics, just as fables, stories, and anecdotes are important to the young in understanding real life'.

Terry Tao

August 2022

Electronic Version of this Newsletter
Email enigma.mensa@yahoo.co.uk and I'll send you a copy

About Enigma
Enigma is the newsletter of Puzzle SIG.
The SIG for anyone interested in puzzles. The scope covers word puzzles, crosswords, logic puzzles, Japanese puzzles, mathematical brain teasers, lateral thinking problems, quizzes and picture quizzes, discussion of physical / mechanical puzzles, computer / internet based puzzles and puzzle games, and puzzle books and publications, and experimentation and innovation of new puzzle forms.

SIGSec and Editor: Elliott Line
34 Hillside, Hartshill, CV10 0NN
www.elliottline.com
enigma.mensa@yahoo.co.uk
Deputy SIGSec: Paul Bostock

How to Join
You can join Puzzle SIG by visiting mensa.org.uk/sigs (member login required).

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Welcome to Enigma 252

Hello and welcome to another issue of Enigma.

Apologies for taking so long once again between issues. The good news is that even over the several months I wasn't working directly on the newsletter I was still creating all sorts of puzzles, so when I did come to put it together I had plenty to work with. It seems I may have gone a little over the top with the solutions though, as they fill more pages of this issue than the actual puzzles! It's useful though to go through the detailed solution even, or maybe especially, if you weren't able to solve the puzzle, to understand how you might have approached it. Some of my favourite YouTube channels: 'Mind Your Decisions', 'Let's Think Critically' and 'Michael Penn' often pose questions which I'm initially clueless about, but I enjoy following along with the solution.

This issue isn't quite all my own work: I am grateful to Paul Bostock and Christa Ramonat for their contributions, which I very much enjoyed solving.

Please try all of the puzzles, and if you have any feedback, let me know and I'll pass it on to the puzzle's creator.
As ever, if you get stuck and need a hint, drop me a line and I'll be happy to help. Please keep your puzzles, answers, comments, queries, suggestions, etc coming in.

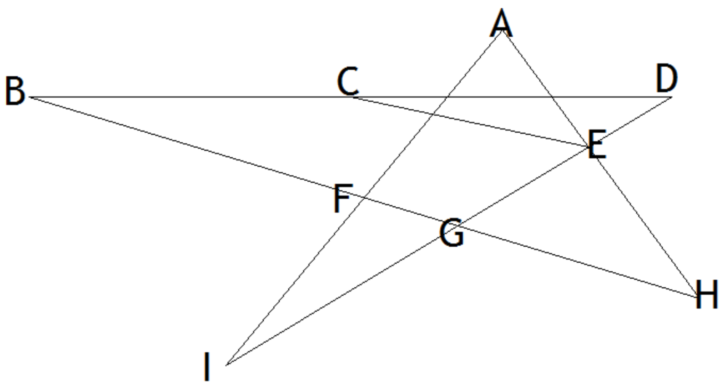
Happy puzzling
Elliott.

251.01 - COMPETITION: More or Less - Elliott Line

x is 74.
The triangle has sides 251, 325 and 177.
The angle opposite the base is 49.9998…

WELL DONE TO:
Johann Muller
Karl Billington
Ivor Cornish
Paul Clark
Matthew Francis
Paul Bostock
Stuart Nelson
Christa Ramonat
Roisin Carters
Angela Richens
Julie Harkin

252.01 - COMPETITION: Starshaped - Elliott Line



In the diagram above, F is the midpoint of both BH and AI.
G is the midpoint of DI, and lies on the line BH
E is the midpoint of AH and lies on the line DI.
C is the midpoint of BD.
BH is precisely 252 long (252 being the number of this issue of Enigma).

What is the length of CE?

This is a competition, but not for prizes, only bragging rights. Every correct answer I receive will get an honourable mention in the next issue of Enigma. Send your answer to me at enigma.mensa@yahoo.co.uk .

252.02 – Alphabet Crossword – Elliott Line

Your task is to reconstruct this crossword. One of each of the 26 letters of the alphabet have been removed from the crossword and needs to be put back in. In addition, the black squares need to be added in.

The final pattern of black squares is symmetrical in the horizontal axis and in the vertical axis, but NOT along the diagonal axes. There is at least one black square in every row and every column, and all words are at least three letters long. Every word in the final solution contains both given (black) letters and alphabet (red) letters.

	U						S	T
U		L						A
		L	E		A	N		
D			L		N			
		E	A			H		
			N		I			S
U		E				L	A	
F		G						A
	I		S			A		

A	J	S
B	K	T
C	L	U
D	M	V
E	N	W
F	O	X
G	P	Y
H	Q	Z
I	R	

252.03 – Come Close But Don't Touch - Elliott Line

If you have five unit fractions (a unit fraction is a fraction with 1 on top and a positive whole number on the bottom, such as 1/2, 1/3 or 1/1000), and you add them together, how close can you get to a sum of 1 without actually equalling 1 or going beyond it?

For instance, if the first four fractions were each 1/5, the most the last one could be is 1/6, which would make the sum 29/30. Close to 1, but not equal, as we are seeking. You can get even closer of course, but you can't get arbitrarily close; there is a maximum.

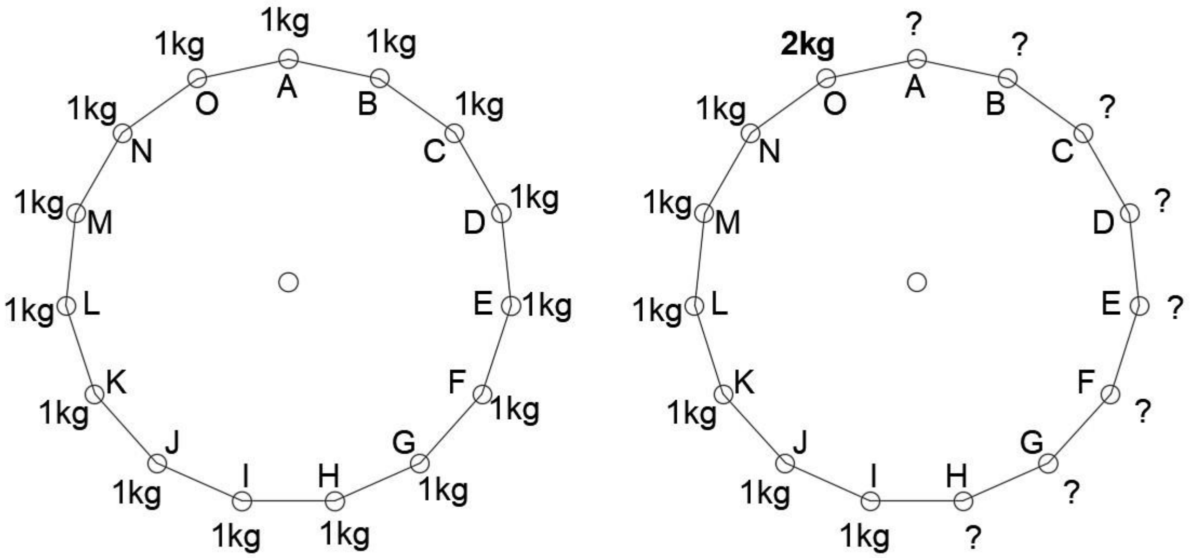
252.04 – Cyberpunk Number - Elliott Line

Consider the letters of the word CYBERPUNK. There are nine letters without repeats, so therefore there are 9! (=362880) different ways of arranging them.
If you place those 362880 ways in alphabetical order and number them: 1=BCEKNPRUY, 2=BCEKNPRYU, 3=BCEKNPURY, … 362880=YURPNKECB, which number is CYBERPUNK?

Bonus question: which string of letters will be 100,000th in the sequence?

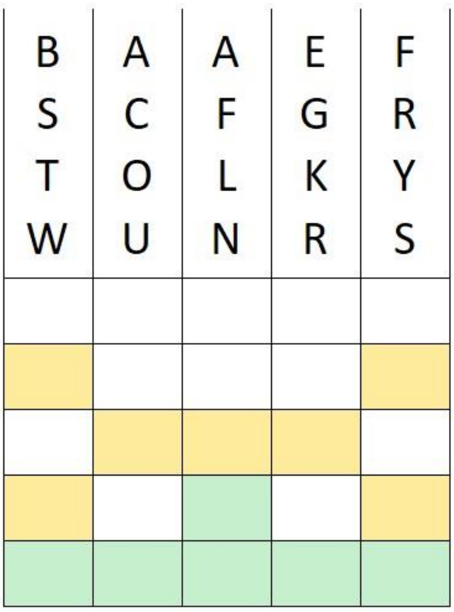
252.05 – Balancing Act – Elliott Line

I start with a regular pentadecagon (15-sided shape) with has a kilogram weight on each of its 15 vertices. Clearly the centre of gravity will be in the exact centre of the shape. Someone comes along with a 16th kilogram weight and places it at vertex ‘O’.
How can I take the eight weights from vertices A to H and redistribute them amongst vertices A to H so that the overall centre of gravity is once more in the exact centre of the shape? The weights cannot be subdivided, and must be placed on those vertices, not elsewhere on the shape.



Bonus question: if you wanted to put a 17th weight somewhere on the left-hand side (positions I to O), such that the eight weights on the right-hand side (A to H) could again be rearranged to balance the system, where would the 17th weight need to be placed?

252.06 – Construct-a-Wordle – Christa Ramonat



In the Wordle left, the letters for the first four guesses are given, and are above the correct column. The puzzle in threefold: firstly reconstruct what the first four guessed words were; secondly, place the four words in the correct order in the grid; thirdly, use the green, yellow and white squares from the first four rows to work out what the actual secret word should be.

252.07 – First Date

If Friday 18th April is at the start, what date is at the end?

252.08 – Eleventh Root - Elliott Line

If I tell you that N is an integer and that N^11 = 2,472,159,215,084,012,303
can you tell me what N is, IN YOUR HEAD, without using any electronic assistance, nor even pen and paper?

It might help if you know that Fermat's Little Theorem states that
n^p (mod p) = n (mod p)

252.09 – Isosceles Nine-point Circle - Elliott Line

It is well known that for any triangle, the following nine points:
a) the three midpoints,
b) the three feet of the altitudes (the altitudes are the lines from each vertex to the opposite side, perpendicular to that side),
c) and the three points midway between the vertices and the orthocentre (the orthocentre is the point where the three altitudes cross),
all lie on a circle, known as the Nine-Point Circle.
Sometimes some of those points coincide with one another, for instance the midpoint of the base of an isosceles triangle is also the foot of the altitude from the apex down to the base.
I have an isosceles triangle, and so since two of the nine points coincide, there are now eight distinct points. In my triangle these eight points are equally spaced, forming the vertices of a regular octagon.

What is the angle at the apex of the isosceles triangle?

For bonus points, what is a second possible value for the apex angle?

252.10 – Isosceles Nine-point Circle Part 2 - Elliott Line

Following on from part 1, I have a similar situation, examining the eight distinct points of a nine-point circle of an isosceles triangle.

However, they now form an IRREGULAR octagon. In particular, the two sides of the octagon that meet at the midpoint of the base are one length, and the other six sides are a second length.

Now what is the angle at the apex of the triangle?

(For uniqueness, let the angle be less than 90 degrees).

252.11 – Letter Swaps – Elliott Line

In each of these three puzzles there is a 4x4 word square, which has four words reading across, and four others reading downwards, except that it is all wrong!
Precisely none of the letters is in the correct place in the grid. However, they have not been placed randomly: eight pairs of letters need to be swapped in order to solve the grid. So, for instance if you decided that the top left letter belongs in the bottom right square, that would also mean that the bottom right letter belongs in the top left square.
I can also tell you that none of the swaps are within a row or column: each letter will end up in a different row AND column from where it started.
Beware of repeated letters: even if you know for sure that a particular letter belongs in a certain square, if there are several copies of that letter in the grid, you have to choose the correct one, otherwise the other letter of the swapped pair would end up in the wrong place.
Because it is very difficult to get started with these puzzles, I will tell you one of the solution words for each puzzle, but you'll have to decide where it goes.
Good luck!

E	E	M	L
A	E	A	A
V	C	O	X
N	T	A	P

starter word: CAVA

A	F	E	D
T	G	L	R
I	E	E	F
T	A	I	N

starter word: IDLE

T	O	S	I
I	G	P	M
L	E	L	E
N	T	P	U

starter word: OMIT

252.12 – Missing Integers - Elliott Line

$2^a + 2^{10} + 2^{13} = b^2$

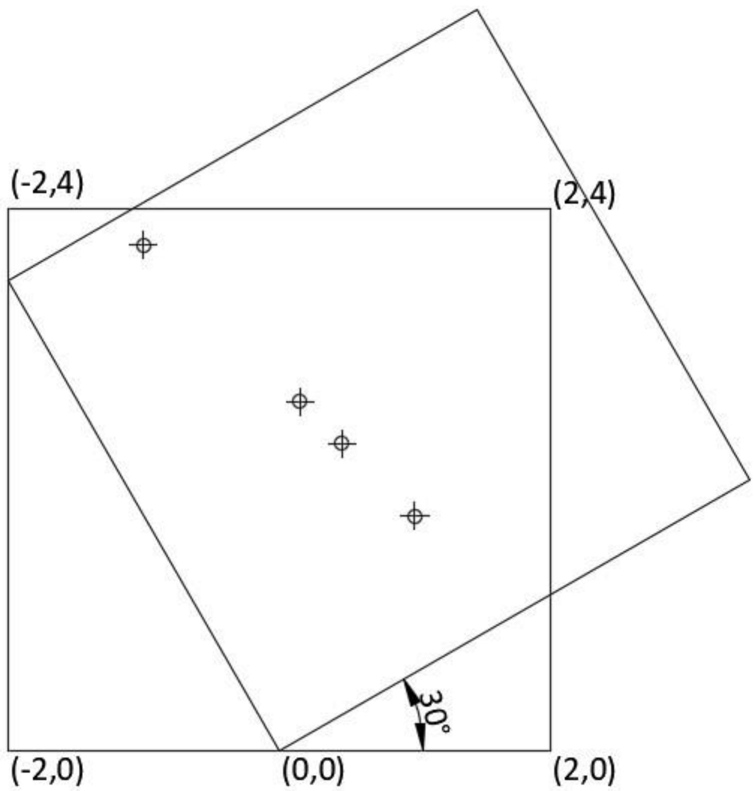
a and b are both natural numbers, what are they?

252.13 – PIN Number – Paul Bostock

I have a four-digit PIN, four different digits in ascending order. The sum of the squares of the first three digits is the square of the fourth digit, but none of the digits is itself a square.
What is my PIN?

252.14 – Points of Rotation - Elliott Line

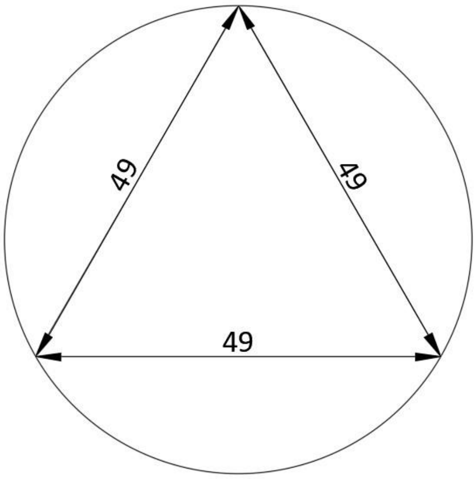
I have two identical 4 x 4 squares as shown, identical except for orientation and position. Clearly it is possible to move the tilted square to the position of the upright square by first moving it so that one of its corners coincides with that of the upright square, and then to perform a rotation so that the other three corners also coincide. You might also choose to perform a rotation first, and then a move to get the squares to line up.
Anyone who's ever studied group theory will know that the result of rotation plus a translation can always be done using ONLY a rotation - it's just a question of figuring out where the centre of rotation needs to be.
Since the shape we are using, a square, has order 4 rotational symmetry itself, there are in fact four possible centres of rotation, depending on which of the four sides of the tilted square eventually coincides with the base of the upright square. For instance, a 30 degree clockwise rotation around the top rotation point will align the squares such that the bottom right side of the tilted square becomes the base, whereas a 60 degree anti-clockwise rotation around the lowest rotation point will also align the squares, but now the bottom left side of the tilted square coincides with the base of the upright square.



As an aside it is interesting to note that these four points are collinear.
This week's challenge is to find the co-ordinates of two of these four centres of rotation. Or for extra kudos, the coordinates of all four.

252.15 – Points on a Circle - Elliott Line

Inscribed within a circle is an equilateral triangle of side length 49.
How many points are there on the circle that are an exact integer distance from all three vertices of the triangle?



252.16 – Power Dates - Elliott Line

Using the DDMMYY format, you can change a date to a six (or sometimes five) digit number, for instance 15^{th} August 2022 would be 150822, and 1^{st} November 2023 would become 011123. Some of these dates can be perfect powers, for instance 11th February 2024 would be 110224, which is 332 squared, and 9^{th} November 2025 would be 091125 which is 45 cubed. You can also make 4^{th} and 5^{th} powers, and also one more beyond that. What is the only date that becomes a power beyond the fifth power?

252.17 – Round and Round - Elliott Line

Choose a value x such that:
You begin with 3x, and round to the closest whole number.
Multiply this by 4x, and round to the closest whole number
Finally multiply this by 5x, and you will have the answer 273.

For example:
Let x = 7/4,
3x = 21/4 = 5.25, round to 5,
Multiply by 4x (7) = 35, round to 35.
Finally multiply by 5x (35/4) to get 306.25
As 306.25 is not 273, this is not the correct value of x. But what is?

252.18 – Twenty Balls - Elliott Line

I asked my 10-year-old son Austin the question: if you had four balls in a bag and you wanted to colour them such that a randomly drawn pair of balls would have a 50% chance of matching in colour, how would you colour them? I was fully expecting him to go for the obvious but wrong answer of two of one colour, two of another. Instead, Austin, quick as a flash, came up with the correct answer of three of one colour and one of a second colour. I was intrigued as to how he did it so quickly; he explained that he visualised the four balls in a 2x2 grid. That way it was obvious to him that with a (3,1) colouring, half the rows half the columns and half the diagonals were matching in colour. Nice one Austin!

I now have 20 coloured balls of various colours in a bag, such that if you pick two balls at random the chance that the two chosen balls match is 50%. For this one there probably isn't a neat visual shortcut, so arithmetic is likely needed.

What does the arrangement of the twenty coloured balls need to be?

252.19 – Totient Trouble - Elliott Line

Allow me to introduce you to Euler's Totient function, phi(n). It is the number of numbers less than a number that don't share any factors with that number. For instance, when n is 6, phi(n) is 2, because there are only 2 numbers less than 6 that are coprime with 6 (1 and 5).
There is a shortcut way of finding the totient function of a number: first list all of the prime factors of the number, then go through them one by one, if you see a prime factor for the first time, subtract 1 from it, but if it's one you've already seen, leave it as is. Then multiply the (some now modified) factors back together. For example, the totient function of 24:

24 = 2x2x2x3, phi(24) = (1)x2x2x(2) = 8.

Now after that crash course it's going to get even more complicated as we consider doing it in reverse. The inverse totient function lists all of the numbers n for which phi(n) equals a particular value. We have seen that phi(24)=8, but for what other values of n is phi(n)=8? The full list is 15, 16, 20, 24 and 30. Only one of these numbers is odd (15). This is no accident, and brings us around (finally!) to the question I want to ask you.

To simplify things slightly, I ONLY want to consider cases where n is odd, and phi(n) is a power of 2, for example:
phi(1)=1, phi(3)=2, phi(5)=4, phi(15)=8, phi(17)=16, phi(51)=32, phi(85)=64, etc.

In each case there is exactly one odd value for which phi(n) is equal to a particular power of 2. However this pattern doesn't last forever, and eventually we will find that some powers of 2 are not the totient function of ANY odd numbers.

What is the first such power of 2?

252.20 – Zero to Pi(ish) - Elliott Line

I discovered an interesting fact the other day, that you can start with 0, and reach ANY positive rational number by selectively using two different actions:
The two actions are as follows:

a) Adding 1, such that x becomes (x+1)
b) Taking the reciprocal, such that x becomes (1/x)

For example, trying to reach 2/5:

starting with 0,
Action a makes it 1
Action a makes it 2
Action b makes it 1/2
Action a makes it 1 1/2
Action a makes it 2 1/2
Finally action b makes it 2/5

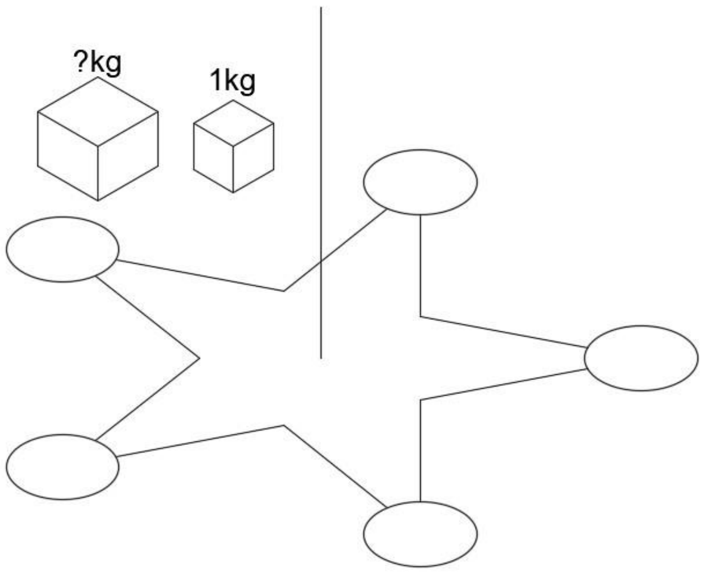
Pi is famously not a rational number, however 355/113 is a great approximation to it. Starting with 0, how many steps will it take to reach 355/113?

252.21 – Star Balance – Elliott Line

You have a supply of two different sizes of blocks: some weigh 1kg, others are heavier than 1kg but not as much as 2kg (you get to decide the exact weight of these heavier blocks but they must all weigh the same).

If none of the five positions is allowed to be left empty, and each of the five piles is different, what is the minimum number of blocks you can arrange at the five points of a regular penta-star so that the entire arrangement is balanced around the centre?

Bonus: would it make a difference if one of the positions can be left empty (but only one as otherwise two 'piles' would be the same)?



252.22 – Unit Fractions - Elliott Line

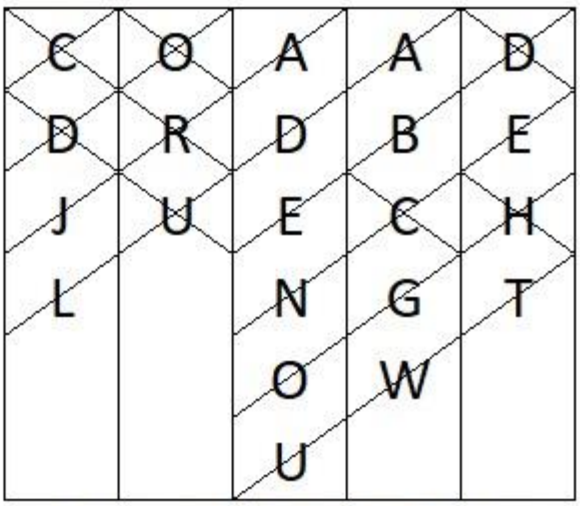
$1/a + 1/b = 5/391$

a and b are positive whole numbers, what are they?

252.23 – Wordsalad - Elliott Line

The object of Wordsalad is to find a set of five-letter words, with the first letter coming from the first column, second letter from the second column etc, that between them use every letter in the grid at least once, but no more than twice.

Example:



COACH, CROWD, DOUBT, DREAD, JUDGE, LUNCH

Puzzle:

A	A	B	I	E
O	L	D	N	O
R	N	G	O	S
	O	I	S	T
	T	O	V	
		S		

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252.02 – Alphabet Crossword – Elliott Line

Q	U	I	Z		J	U	S	T
U		L				R		A
A	X	L	E		A	N	O	N
D			L		N			K
	H	E	A	L	T	H	Y	
B			N		I			S
U	S	E	D		C	L	A	W
F		G				E		A
F	I	G	S		V	A	M	P

252.03 – Come Close But Don't Touch - Elliott Line
It turns out the best approach is a kind of 'greedy' algorithm, where each unit fraction is the most it can be, given how much is left of the 1 we are trying to almost fill. This is easily calculated: if there is 1/n remaining, the next biggest unit fraction will be 1/(n+1).
So the first unit fraction will be 1/2, leaving 1/2.
Therefore the second unit fraction is 1/3, leaving 1/6.
Next is 1/7, leaving 1/42.
Next is 1/43, leaving 1/1806.
Finally we add 1/1807, leaving us just 1/3263442 short of the 1.
So our final sum comes to 3263441/3263442 = 0.9999996936…

252.04 – Cyberpunk Number – Elliott Line
The basic way to solve this is to realise that the first 1/9 of letter strings start with B, the next 1/9 start with C, etc. And then within a 1/9 section, the first 1/8 start with the first available letter, and so on.
The most intuitive approach is to discard all those strings that come before CYBERPUNK, counting the discarded strings as we go.
So firstly, because C is 2^{nd} alphabetically, we discard any strings that start with B, which is 1/9 of them, or 40320 (1/9 of 9! is simply 8!). Of the remaining letters Y is 8^{th}, so within the 1/9 that start with C, we want to discard the first 7/8 of them, or 35280 (which is 7*7!), etc.
Mathematically how this works is to express the word CYBERPUNK with numbers detailing where each letter comes alphabetically out of the letters remaining at that point. So C is second out of BCEKNPRUY. Then Y is 8^{th} out of BEKNPRUY, B is first out of BEKNPRU, E is first out of EKNPRU, etc. This gives (2,8,1,1,4,3,3,2,1).
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Because we want to discard all of the strings before CYBERPUNK, we subtract 1 from each of those numbers to get (1,7,0,0,3,2,2,1,0), and then we need to multiply each of those numbers by the appropriate factorial, and add them together:
1*8!+7*7!+0*6!+0*5!+3*4!+2*3!+2*2!+1*1!+0*0!
40320+35280+0+0+72+12+4+1+0=75689.
But remember, this is the number of strings before CYBERPUNK, so the actual CYBERPUNK number is the next one: 75690.

To answer the bonus question, we need to the same thing but in reverse. To start with we need to subtract 1 to get 99,999. We then need to express that as the sum of factorials 8! to 0! : (2*8! + 3*7! + 5*6! + 5*5! + 1*4! + 2*3! + 1*2! + 1*1! + 0*0!). Then we need to add 1 to each of these: (3,4,6,6,2,3,2,2,1). Finally, we write out the nine letters in alphabetical order: BCEKNPRUY, and the first of our word string will be the 3rd in alphabetical order: E. Then second in our string will be the 4th in alphabetical order (of those letters than haven't been used yet), etc. The 100,000th string will therefore be ENUYCPKRB.

252.05 – Balancing Act – Elliott Line

It is tempting to think you can exactly mirror the arrangement of weight on the left-hand side, so to have 0 at A, 2kg at B and 1kg elsewhere, however this gives a centre of gravity that is a little way above the centre of the shape, slightly in the direction of A. Instead, we can use the fact that 15 is the product of 5 and 3. A subset of five weights equally spaced (on a regular pentagon) around the shape will have a centre of gravity at the centre of the shape. Likewise, three weights, equally spaced (on an equilateral triangle) will also be balanced. We are looking to place 16 weights altogether, which suggests we can divide it into two pentagons and two triangles, since 5+5+3+3=16. Since we know O has 2kg, we can place both a pentagon and a triangle on that vertex, but that leaves K, M and N short of a weight. We can then place a pentagon through K and N, and a triangle through M. This completes the given weights on the left-hand side, and also determines the positions of the other eight weights on the right-hand side, such that A, D and G have no weight, B and F have a 1kg weight, and C, E and H have 2kg.
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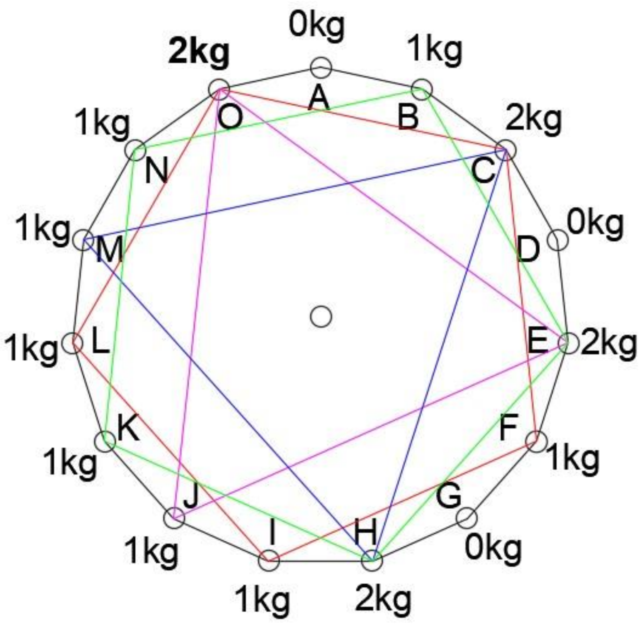

As for the bonus question, with now 17 weights in total, the only way to make this a sum of 3s and 5s is 5+4*3. The only way to do this with five of the positions I to O containing 1kg is to have the pentagon CFILO and the four triangles EJO, AFK, CHM and DIN. This will rearrange the 8 weights on A to H as 10211201, and the positions I and O will have the extra weights on the left-hand side.

252.06 – Construct-a-Wordle – Christa Ramonat

B	U	L	K	Y
T	O	N	G	S
W	A	F	E	R
S	C	A	R	F
F	E	A	S	T

252.07 – First Date

Of the days of the week, Friday is first in alphabetical order, and amongst the months April is first. When the date is written out in full, 'eighteenth' is first alphabetically.

The final day, date and month alphabetically will respectively be Wednesday, twenty-third and September, so the answer is Wednesday 23rd September.

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252.08 – Eleventh Root - Elliott Line

While the question says that it must be solved in your head, I'm obviously having to present it here on paper, but hope you'll agree that the individual steps can be done mentally.

First of all we want to know how many digits in N. If N was 10, N^11 would be 1 followed by eleven zeroes. If N was 100, N^11 would be 1 followed by twenty-two zeroes. Since the N^11 in the question has 19 digits, N must be between 10 and 100, and so therefore has two digits.

Let's find the last digit first. The final digit of N^11 is 3. Clearly the digit we seek cannot be even. We can try out each of the odd digits. To do this we can repeatedly multiply by a digit and discard all but the final digit, until we get back to where we started. Powers of 5 always end in 5. Powers of 1 always end in 1. Powers of 9 alternate between 9 and 1. So we are left with 3 or 7. 3 follows the repeating pattern 3 - 9 - 7 - 1, and 7 follows the repeating pattern 7 - 9 - 3 - 1, so both could end in a 3. But we are looking for an eleventh power, and since both these sequences are four in length, it is the third in the repeating sequence that we are interested in. Therefore the final digit of N is 7.

Next, since 11 is prime, we can make use of Fermat's Little Theorem to find out the divisibility of N with respect to 11. The test for divisibility by 11 is very easy: add and subtract alternate digits, and if the result is a multiple of 11, then the original number was too. Determining modulo 11 is similar but now we must make sure that the final digit is added, so in other words all odd-positioned digits need to be added and all even-positioned digits subtracted.

Working through the given number in order:

2-4+7-2+1-5+9-2+1-5+0-8+4-0+1-2+3-0+3 = 3

Since N^11(mod 11) = N(mod 11), we know that N must be three more than a multiple of 11. Since we already know the final digit is 7, we can tell that the first digit is 4.

Therefore N = 47.
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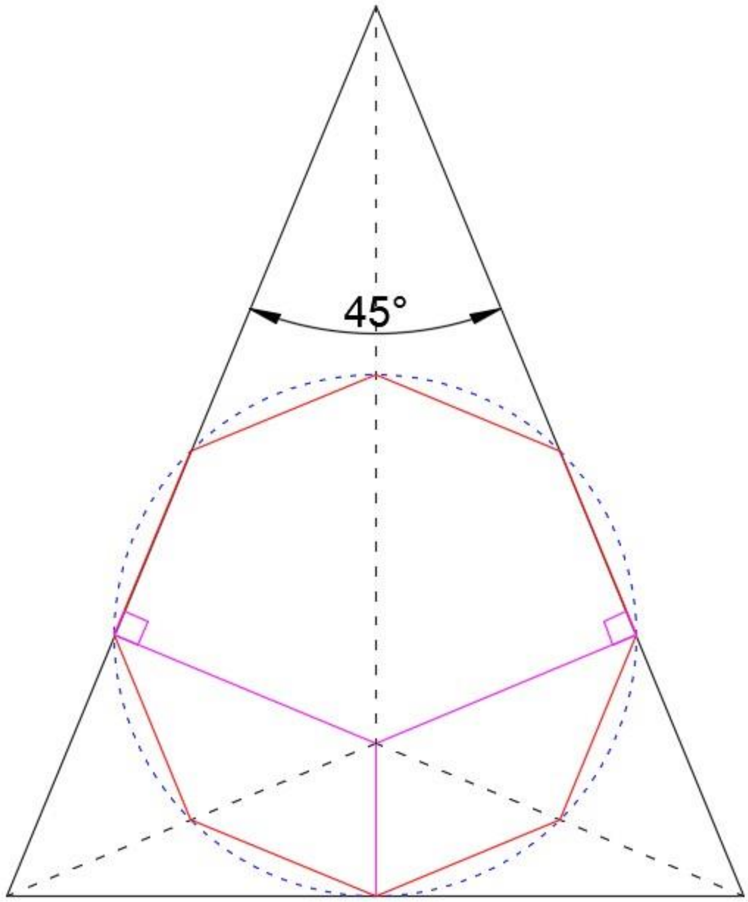
252.09 – Isosceles Nine Point Circle - Elliott Line



The angle at the apex will be 45 degrees.
I alluded to the existence of a second possible solution. An interesting fact is that if you consider the orthocentre of any triangle along with the three vertices, you can in fact choose any three of those four points to form a triangle, and the resulting nine point circle will be the same, with the nine points in all the same places (although midpoints might become 'halfway to orthocentre' points and vice versa). So this means there are four possible triangles with the exact same set of eight points as above. However, since we are specifically asked for an isosceles triangle, there is only one additional triangle which fits the bill, ie the triangle formed by the two lower points of the above triangle, along with what was the orthocentre. This has an apex angle of 135 degrees.
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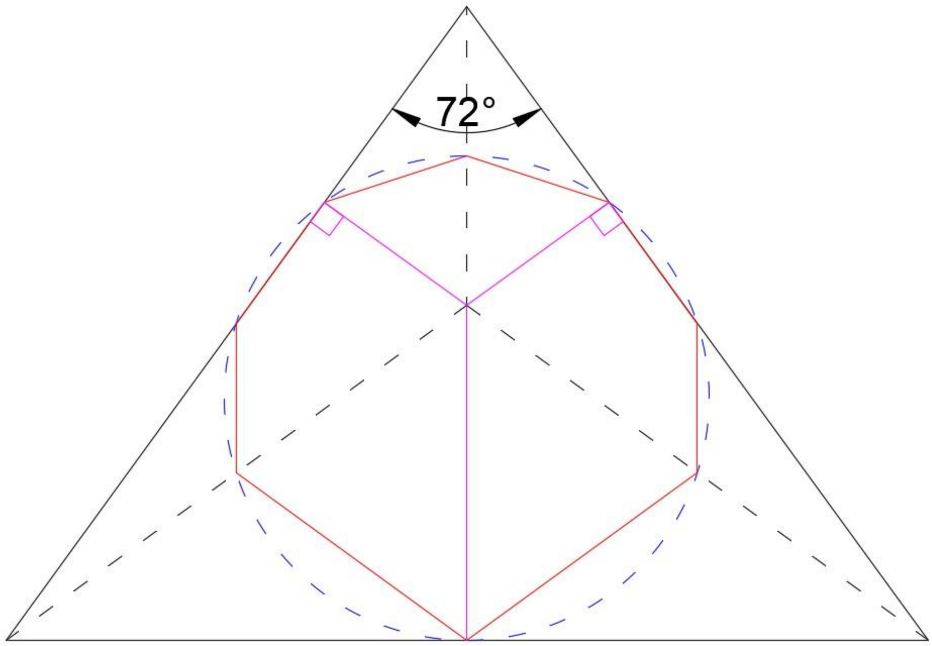
252.10 – Isosceles Nine Point Circle Part 2 - Elliott Line



The angle at the apex will be 72 degrees.
The two longer lengths are two sides of a regular pentagon, and the six shorter sides are sides of a regular decagon.
The ration of long length to short length is sin(36)/sin(18) = sqrt(1/2(5+sqrt5)) ~ 1.902…

252.11 – Letter Swaps – Elliott Line

E 1 C	E 2 A	M 3 V	L 4 A
A 5 O	E 6 P	A 2 E	A 7 N
V 3 M	C 1 E	O 5 A	X 8 T
N 7 A	T 8 X	A 4 L	P 6 E

A 1 R	F 2 I	E 3 N	D 4 G
T 5 I	G 4 D	L 6 E	R 1 A
I 2 F	E 6 L	E 7 A	F 8 T
T 8 F	A 7 E	I 5 T	N 3 E
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T 1 P	O 2 L	S 3 U	I 4 G
I 5 L	G 4 I	P 6 N	M 7 E
L 2 O	E 7 M	L 5 I	E 8 T
N 6 P	T 8 E	P 1 T	U 3 S

252.12 – Missing Integers - Elliott Line

2^a + 2^10 + 2^13 = b^2

Looking first just at 2^10 + 2^13, since 10 is less than 13 we can take out a factor of 2^10 to get:

2^10(1 + 2^3)

2^10 is obviously (2^5)^2 and (1 + 2^3) is 9, also a square number, therefore, you can replace 2^10 + 2^13 by (3*2^5)^2, or 96^2

2^a + 96^2 = b^2

Moving this to the right-hand side we have:

2^a = b^2 - 96^2

Factoring the difference of two squares gives:

2^a = (b+96)(b-96)

This means that both of the factors on the right-hand side are powers of 2, let’s call them 2^c = b+96 and 2^d = b-96.

If we take their difference to eliminate b we get:

2^c - 2^d = 192

Since d is smaller than c, we can take out a factor of 2^d:

2^d(2^(c-d)-1)=192

which is a product of a power of 2 and an odd number, which must therefore be 2^6 and 3.

d is therefore 6, and c is 8. a, as the sum of c and d, is therefore 14, and b is 2^6 + 96 = 160.

So the completed equation is

2^14 + 2^10 + 2^13 = 160^2

252.13 – PIN Number – Paul Bostock

2367

The squares are 4, 9, 36 and 49 respectively, and 4+9+36=49
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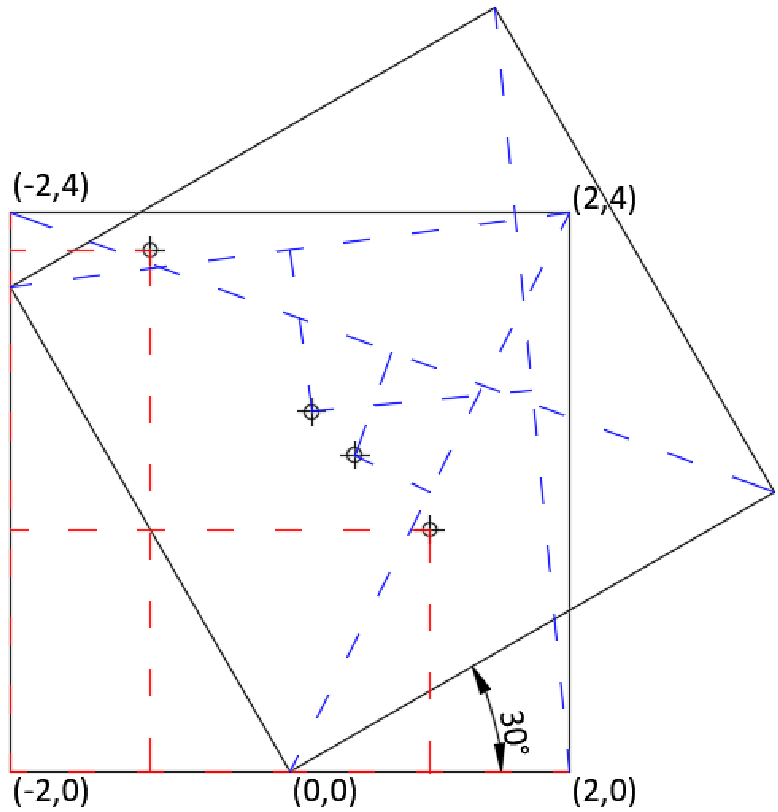
252.14 – Points of Rotation - Elliott Line

If you draw a lines connecting at least two points on the tilted square with the corresponding points on the upright square, then find the perpendicular bisectors of those two lines, then the point that those perpendicular bisectors cross will be in the exact same place in both shapes, and can therefore be used as the centre of rotation of one square to the position and orientation of the other. There are two obvious ways to do this in the figure so I would regard these as the two easier solutions to find; since the connecting lines are horizontal and vertical, the perpendicular bisectors will be vertical and horizontal.
The lower point has coordinates (1, sqrt(3)), and the upper point has coordinates (-1,2+sqrt(3)).
The other two points can be found using the same method, but now the perpendicular bisectors are not conveniently parallel to the axes, and so the maths is not so straightforward. Nevertheless, the coordinates of the points can be determined to be: (2*sqrt(3)/3-1,sqrt(3)/3+2) and (2*sqrt(3)-3,4-sqrt(3)).
Numerically, to three decimal places, the four points are, from top to bottom: (-1,3.732), (0.155,2.577), (0.464,2.268) and (1,1.732).
Of course, there's no particular need for the two corresponding points that you choose to connect, to be corners of the square. If instead you chose the exact centre of the two squares, and construct the perpendicular bisector as before, the rotation point will also lie on this line, and since the four different rotation options will all have the same position as the centre of the square, that explains why the four points are collinear.



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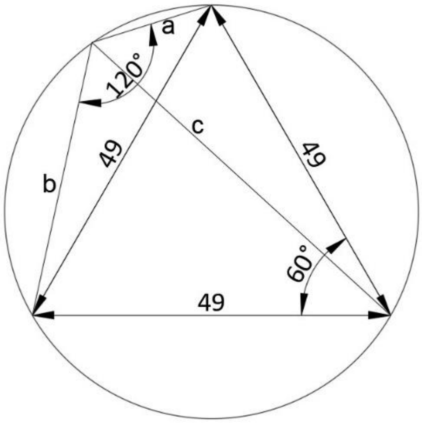
252.15 – Points on a Circle - Elliott Line



We can start off by saying that c>=b>=a, and worry about reflections and rotations later.
First, we'll deal with situations where some of the lengths are equal. Clearly they can't all be equal, as the point would be the centre of the circle, not on it. If a=b, then c would be a diameter of the circle, which is 49*(2/rt(3)) ~ 56.5, not an integer. If b=c, then a=0 and we have a vertex of the original triangle. There are three vertices, so already three points which satisfy the question.

We can now make use of a couple of facts about cyclic quadrilaterals: by Ptolemy's theorem 49c = 49a+49b, so therefore a+b is always equal to c, which is convenient since if a and b are both integers, c will be too. Because opposite angles of a cyclic quadrilateral sum to 180, the angle between a and b is 120 degrees. Now using the fact that cos(120) = -0.5, we can use the cosine rule to find the following equation:

49^2 = a^2 + b^2 + ab

If we add ab to both sides, then factorise the right-hand side we get:

49^2 + ab = (a+b)^2

a+b=c must be between 49 and the diameter ~56.5, so there are only 7 possible integer values to try, and each will give a different value for ab

(a+b,ab) =
(50,99),(51,200),(52,303),(53,408),(54,515),(55,624),(56,735)

Only the last two of these have solutions in the integers:
(a,b,c) = (16,39,55) and (21,35,56).
In our diagram a is the distance from the top vertex, b from the left vertex and c from the right vertex, but these vertices can be permuted in 3! = 6 ways, so these two numerical solutions represent 12 points on the circle. Along with the three vertex points themselves there are therefore 15 points on the circle that are an integer distance from all three vertices of the triangle.
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252.16 – Power Dates - Elliott Line
The only date that's a power higher than 5^{th} is Thursday 13^{th} October 2072, which becomes 131072, which is 2^17.

252.17 – Round and Round - Elliott Line
We can get an approximate value for x by ignoring the rounding steps temporarily.
5*4*3*x^3 ~ 273
x^3 ~ 4.55
x ~ 1.657…

We know that the round(4x(round(3x)) part has been rounded to an integer, so to try to discover exactly what integer it is we can plug our approximate value of x to the outside of it:

5(1.657)(integer) ~ 273
integer ~ 273/(5(1.657)) ~ 32.95

This is close to an integer, 33, but we need to test it by plugging it back in:

Assume the value of round(4x(round(3x)) is indeed 33:

5x*33=273

x = 273/(5*33) = 273/165 = 91/55

Plugging that value into round(4x(round(3x)) indeed gives 33, so this is the correct answer.

x = 91/55 = 1.6545454..

Since this is a strictly increasing function (except where 3x rounds to zero), once we have found an answer, we can be sure it is the only answer.

252.18 – Twenty Balls - Elliott Line
Instead of thinking about probability, think in terms of how many different pairs are possible. Since each pair is equally likely, counting the pairs and comparing them is all we need to do.
There are 20 balls altogether, so there are 20 options for the first ball. There are 19 options for the second ball, but since a pair is the same pair if they are drawn in reverse order, we can halve this product. So the total number of possible pairs is 20 x 19 / 2 = 190.

We can perform the exact same calculations for each set of same-coloured balls within the bag. For instance if there were 15 red balls, then there are 15 x 14 / 2 = 105 pairs that would be made up of two red balls. This is already more than half of 190, so there cannot be 15 balls of a particular colour.
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Instead let's try 14 red balls. This gives us 14 x 13 / 2 = 91 red-red pairs. This is close to the 190 / 2 = 95 pairs we need, so we are on the right track.

If there were 3 balls of a second colour (say, blue) then there would be 3 x 2 / 2 = 3 possible blue-blue pairs. Add these to the 91 red-red pairs and we are almost there.

Add in 2 balls of a third colour (say, white) then there is one white-white pair. This makes up all of the 95 pairs we needed, but we have only used 19 balls, so we need a single ball of a fourth colour (say, green) to make it up to 20.

14 RED, 3 BLUE, 2 WHITE, 1 GREEN.

If you play about with other possibilities you will find that this arrangement, (14,3,2,1), is the only way of exactly achieving the aim of exactly 50% chance of a matching pair.

252.19 – Totient Trouble - Elliott Line

The key to this sequence, and you may have spotted it in the first few terms I gave in the question, are the Fermat numbers: $F_n = 2^{2^n} + 1$. The first few Fermat numbers are 3, 5, 17, 257, 65537. All five of those numbers are prime, so the totient function of any of them, or any product of a combination of them is as follows:

$\phi(F_a \times F_b \times F_c) = 2^{2^a} \times 2^{2^b} \times 2^{2^c} = 2^{(2^a + 2^b + 2^c)}$.

These first five values cover the subscripts 0 to 4, so by using the properties of binary numbers, we can produce an odd number with a totient function of the for 2^k for any k that can be made by combining different powers of 2.

To demonstrate, a selected few are as follows:

$1 = 2^0$
$2 = 2^1$
$3 = 2^1 + 2^0$
…
$7 = 2^2 + 2^1 + 2^0$
$8 = 2^3$
…
$30 = 2^4 + 2^3 + 2^2 + 2^1$
$31 = 2^4 + 2^3 + 2^2 + 2^1 + 2^0$

For example, for k=7, the totient function of the number $F_2 \times F_1 \times F_0 = 17 \times 5 \times 3 = 255$ is $2^{(2^2 + 2^1 + 2^0)} = 2^{(4+2+1)} = 2^7$.

So this can give us any power of 2 as a totient of an odd number, as long as the Fermat numbers are prime, however as suggested in the question, this doesn't continue forever. This is because the next Fermat number F_5 = 4294967297 = 641 x 6700417 and as such is not prime. So when we try to use this method to reach 2^32, the totient function of 4294967297 is not the desired 2^32 = 4294967296, but is in fact 640 x 6700416 = 4288266240. In fact because there are no more known Fermat primes, beyond the first 31 there are no more powers of 2 known to be the totient of an odd number.

252.20 – Zero to Pi(ish) - Elliott Line

The easiest thing is to do the whole procedure in reverse. Action a then become x-1 (action b is unchanged).
So starting with a rational number, to get to 0 in the fewest steps, whenever your number is greater than or equal to 1, perform action a and whenever your number is less than 1, perform action b.
In other words, each time the numerator is less than the denominator, flip the fraction upside down, and in doing so you will successively reduce the denominator until it is 1 and you can then keep subtracting until you get to zero.

From 355/113 to 0 will take a total of 28 actions:

355/113 = 3 16/113
a, 3 times
16/113
b
113/16 = 7 1/16
a, 7 times
1/16
b
16
a, 16 times
0

Reversing the process to go from 0 to 355/133 will clearly also take 28 actions: 16a,b,7a,b,3a.

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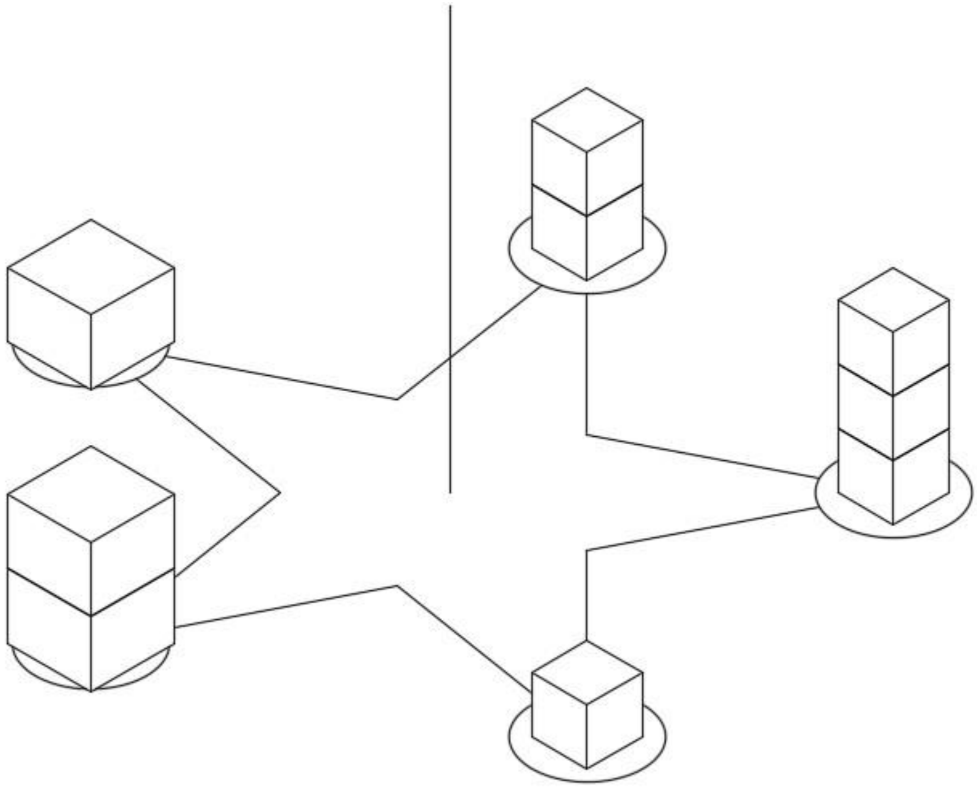
252.21 – Star Balance – Elliott Line

Unlike we saw with the 15-sided shape a couple of weeks ago, 5 is prime and so cannot be broken down into products. But since the five piles need to be different, we do still need to find a way of placing a subset of weights such that they balance. One way would be to have 1kg blocks at two adjacent vertices, and then to calculate what the opposite vertex would need to be to balance.

We can call the distance from the middle to each of the vertices 'd'. Two 1kg weights on adjacent vertices 'd' from the middle would be equivalent to 2kg midway between those two points. That midway point is about 0.8*d from the centre. To balance this with a single weight 'd' from the centre on the opposite side, the distance multiplied by the weight needs to come to the same value. 0.8d x 2kg = d x 1.6kg. This is between 1kg and 2kg as the question dictates. A bit more precise calculation and this balancing block needs to weigh 'phi' kg. Phi is the golden ratio, equal to (sqrt(5)+1)/2 or ~1.618.

So, we have an arrangement which balances with three blocks: one at ~1.618kg and two at 1kg. How many of these three-block arrangements do we need to make the five piles all different? We can do it with just three of them. Place 1 heavy block on a vertex, 2 heavy blocks to an adjacent vertex, and the 1kg blocks required to balance them will be in piles of 1, 3 and 2.

So, we have achieved a balance using nine blocks in total: six 1kg blocks and three ~1.618kg blocks. The weight at each vertex is: ~1.618kg, ~3.236kg, 1kg, 3kg, 2kg.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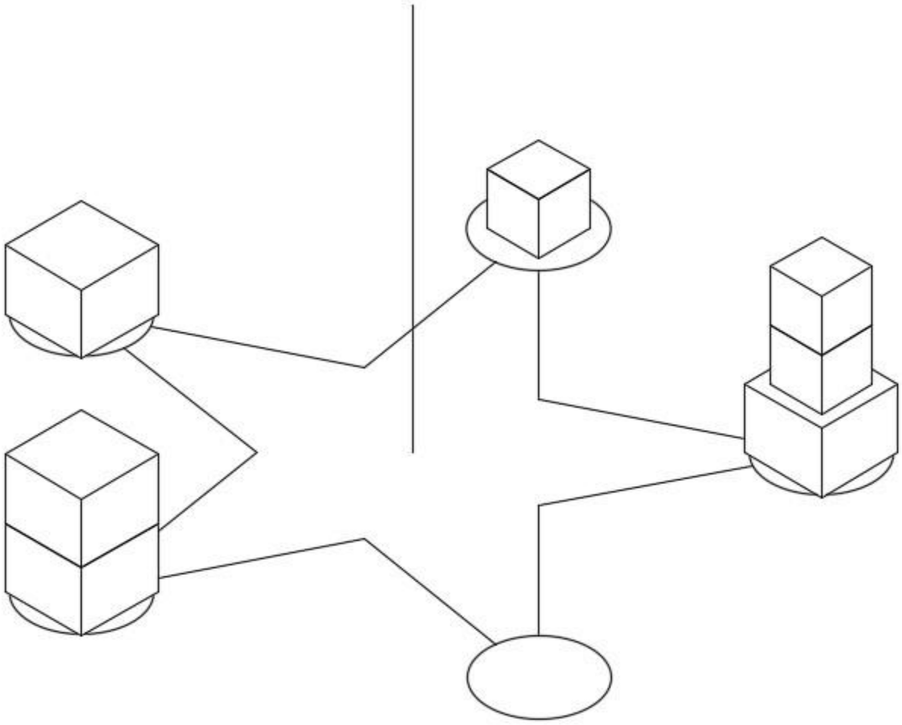
How about now, we see what happens if we place two of the heavy weights on adjacent vertices and calculate what we would need in the single opposite vertex to rebalance the star? Since the pair of weights are scaled up by phi from our original (1,1,phi) subset, the balancing vertex will also be, and will equal (phi)^2. But because phi is such a special number, (phi)^2 is equal to phi+1, which we can achieve by using one of each of our two weights. Now if we combine one (1,1,phi) subset, and one (phi,phi,phi+1) subset, we can have a different weight on each vertex and balance the star, but this time only using seven weights in total, three of 1kg and four of ~1.618kg. The weight at each vertex is: ~1.618kg, ~3.236kg, 0kg, ~3.618kg, 1kg.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252.22 – Unit Fractions - Elliott Line

1/a + 1/b = 5/391
Change the left-hand side to a single fraction
(a+b)/ab = 5/391
Cross-multiply
391a+391b = 5ab
Move everything to one side
5ab-391a-391b=0
Multiply everything by 5 (so that the coefficient of ab is a square)
25ab-1955a-1955b=0
Add 391^2 to both sides so that the left-hand side can be factorised
25ab-1955a-1955b+391^2=391^2
Factorise the left-hand side
(5a-391)(5b-391)=391^2
Since a and b are positive integers, each of the factors on the left is one less than a multiple of 5, so we must also split the right-hand side into factors that are one less than a multiple of 5. As there are no even factors, this effectively means factors ending in 9. 391 is 17x23, so 391^2 is 17x17x23x23. Numbers ending in 3, when repeatedly multiplied by themselves, end in 1,3,9,7,1,3,9,7, etc. Likewise numbers ending in 7, when repeatedly multiplied by themselves, end in 1,7,9,3,1,7,9,3, etc, moving the opposite way around the same cycle. So the only factors ending in 9 from 17x17x23x23 are 17x17 and 23x23. These equate to 289 and 529.
Let 5a-391=289 and 5b-391=529.
This gives a=136 and b=184. We could just as acceptably assign the factors the other way round to get a=184 and b=136, but these are the only solutions.
Finally, to check:
1/136+1/184 = 23/3128+17/3128=40/3128=5/391

252.23 – Wordsalad - Elliott Line

ANGST, ATONE, OASIS, OLIVE, RADIO, ROBOT.
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